Boğaziçi University
Department of Mathematics
M.S. Program Entrance Exam
Time: 9:00 – 12:00

Name: ______________________________

PART A: Answer only **two** of the four questions below.

A 1 Determine whether each of the following expressions converges or diverges. Justify your answers. Indicate the theorems you use (if any).

(a) $\lim_{n\to\infty}\left(1+\sin\left(\frac{1}{n}\right)\right)^n$

(b) $\sum_{n=2}^{\infty}\frac{1}{n\log n}$

(c) $\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$

A 2 Evaluate the following integrals.

(a) $\int_0^1 \sin(\log x)dx$

(b) $\int_0^\infty\int_0^\infty \frac{1}{(1+x^2+y^2)^2}dxdy$

(c) $\oint_{\mathcal{C}}(e^{x^2}-2y)dx+(\log y)dy,\quad \mathcal{C}=\{(x,y)\in\mathbb{R}^2 : (x-2)^2+(y-1)^2=1\}$ with positive orientation.

A 3 Consider the plane $2x+y+z+2=0$ and the paraboloid $z=x^2+2y^2$.

(a) Prove that they are disjoint.

(b) Determine the shortest distance between them. (Hint: It might be helpful to think in terms of tangent planes of the paraboloid.)

A 4 For every $k,n\in\mathbb{N}$ such that $k\le n$, the following identity holds:

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}. \qquad (1)$$

(a) Give two proofs of identity (1):

(i) using a combinatorial argument, and
(ii) using algebraic manipulation.

(b) Prove that the identity

$$\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}$$

holds for every $n,r\in\mathbb{N}$. (Hint: Use identity (1).)

PART B: Answer only **two** of the four questions below.

B 1 Let V be a finite dimensional vector space over $\mathbb{C}$ and let T be a linear operator on V. Suppose that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Prove that the range and the null-space of T are disjoint.

B 2 Let $\mathrm{GL}_n(\mathbb{C})$ denote the group of invertible $n \times n$-matrices with complex entries. Let $M \in \mathrm{GL}_n(\mathbb{C})$ be a matrix of finite order. Prove that M is diagonalizable.

B 3 Let G be a finite group and $g \in G$ be an element of order at least 3. Let $C_G(g)$ denote the centralizer of g in G and suppose that $|G : C_G(g)|$ is odd. Prove that g is not conjugate to g^{-1}.

B 4 Let G be a group and $a \in G$. Prove that a commutes with any of its conjugates if and only if a belongs to an abelian normal subgroup of G.

PART C: Answer only **two** of the four questions below.

C 1 Let $f_n : (0,1) \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to some function $f : (0,1) \to \mathbb{R}$. Prove or disprove each of the following statements.

(a) If f_n converges uniformly to f, then f is continuous.

(b) If f is continuous, then f_n converges uniformly to f.

C 2 Let $f : (0,1) \to \mathbb{R}$ be a differentiable function. Prove or disprove each of the following statements.

(a) If f is uniformly continuous, then f' is bounded.

(b) If f' is bounded, then f is uniformly continuous.

C 3 Let (M,d) be a metric space, $A \subset M$ be nonempty, $x \in A$, and $B(x,r) = \{y \in M : d(x,y) < r\}$ for every $r > 0$. Prove or disprove each of the following statements.

(a) If A is closed and $A \subset B(x,r)$ for some $r > 0$, then A is compact.

(b) If A is compact, then A is closed and $A \subset B(x,r)$ for some $r > 0$.

C 4 Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Suppose f is bounded by a constant M on the circle $\{z \in \mathbb{C} : |z| = R\}$ for some $R > 0$. Prove that the coefficients C_k in the power series expansion of f about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

(b) Suppose there exist real constants A, B and an integer $n \geq 0$ such that $|f(z)| \leq A + B|z|^n$ for every $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most n. (Hint: Use part (a) to control the coefficients C_k for $k > n$.)