Boğaziçi University
Department of Mathematics
Ph.D. Program Entrance Exam
Time: 9:00 – 12:00

Name: ______________________

PART A: Answer only **three** of the four questions below.

A 1 Show that if G is a non-abelian simple subgroup of S_n then G is contained in A_n.

A 2 Let R be a commutative ring with 1 in which every ideal is prime. Prove that R is a field.

A 3 Let R be a ring and $f : \mathbb{Q} \to \mathbb{R}$ and $g : \mathbb{Q} \to \mathbb{R}$ be ring homomorphisms. Show that if $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$ then $f = g$.

A 4 Let W be the space of $n \times n$-matrices over a field F and let f be a linear functional on W such that $f(AB) = f(BA)$ for every $A, B \in W$. Show that f is a multiple of the trace functional.

PART B: Answer only **three** of the four questions below.

B 1 Let (M, d) be a metric space, $A \subset M$ be nonempty, $x \in A$, and $B(x, r) = \{y \in M : d(x, y) < r\}$ for every $r > 0$. Prove or disprove each of the following statements.

(a) If A is closed and $A \subset B(x, r)$ for some $r > 0$, then A is compact.

(b) If A is compact, then A is closed and $A \subset B(x, r)$ for some $r > 0$.

B 2 Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (i.e., f is analytic on the whole complex plane).

(a) Suppose f is bounded by a constant M on the circle $\{z \in \mathbb{C} : |z| = R\}$ for some $R > 0$. Prove that the coefficients C_k in the power series expansion of f about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

(b) Suppose there exist real constants A, B and an integer $n \geq 0$ such that $|f(z)| \leq A + B|z|^n$ for every $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most n. (Hint: Use part (a).)

B 3 Let m denote the Lebesgue measure on the real line, $f : \mathbb{R} \to \mathbb{R}$ be an integrable function and $F(x) = \int_{-\infty}^{x} f \, dm$ for every $x \in \mathbb{R}$. Prove or disprove each of the following statements. Indicate the theorems you use (if any).

(a) F is continuous at every $x \in \mathbb{R}$.

(b) F is differentiable at every $x \in \mathbb{R}$.

(c) F is differentiable at m-a.e. $x \in \mathbb{R}$.

B 4 Let $(X, \mathcal{F}, \mu)$ be a measure space and $(f_n)_{n \geq 1}$ be a sequence of real-valued measurable functions on X. Prove or disprove each of the following statements.

(a) If $f_n \to 0$ in μ-measure, then $f_n \to 0$ μ-a.e.

(b) If $f_n \to 0$ μ-a.e., then $f_n \to 0$ in μ-measure.

(c) If $\mu(X) < \infty$ and $f_n \to 0$ μ-a.e., then $f_n \to 0$ in μ-measure.